A NOTE ON THE ROECKER-RICCHIAZZI MODEL OF PENETRATOR TRAJECTORY INSTABILITY

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Roecker and Ricchiazzi's model of trajectory instability of a penetrator in a soft target is reviewed and analyzed. They considered only a limited set of initial conditions of penetrator yaw and yaw rate. Here we consider a variety of initial conditions, for which it is shown that the model predicts several phenomena associated with trajectory instability, depending on those conditions. Specifically, the bullet may penetrate for some distance before instability becomes evident, it may continuously tumble, or it may reverse orientation and then penetrate stably for some distance. It is also shown that, for small initial yaw and yaw rate, the yaw vs. depth curves for various initial conditions can be superimposed by shifting them along the penetration-depth axis; this validates the method used by Roecker and Ricchiazzi in consolidating their data along a single curve.

INTRODUCTION

Roecker and Ricchiazzi (1978) derived an equation to model yaw growth by penetrators in dense fluids,

$$\frac{d^2\alpha}{ds^2} = M \cos\alpha \sin\alpha \tag{1}$$

where α is the yaw, s is the penetration path length (in calibers), and M is a constant. They gave no solution of this nonlinear equation but observed that, for small values of yaw, for which sin $\alpha \approx \alpha$ and cos $\alpha \approx 1$, it can be approximated by

$$\frac{d^2\alpha}{ds^2} = M\alpha \tag{2}$$

This linear equation has the solution

$$\alpha = \alpha_{01} \exp\left[\sqrt{M}(s - s_0)\right] + \alpha_{02} \exp\left[-\sqrt{M}(s - s_0)\right] \tag{3}$$

where α_{01} and α_{02} are constants. Since the second term vanishes very rapidly, they proposed the solution using only the first,

$$\alpha = \alpha_0 \exp\left[\sqrt{M}(s - s_0)\right] \tag{4}$$

where $\alpha_0 = \alpha_{01} = \alpha$ at $s = s_0$. Then they evaluated the first derivative of this solution, $d\alpha/ds = \sqrt{M}\alpha_0 \exp[\sqrt{M}(s - s_0)]$, at the initial location $s = s_0$ to conclude that

$$\left.\frac{d\alpha}{ds}\right|_{s=s_0} = \sqrt{M}\alpha_0 \tag{5}$$

They stated that instead of three unknowns, M, α_0, and $d\alpha/ds$ at $s = s_0$, only two remain, M and α_0. (M was considered unknown since it was adjusted to fit the data.)

At first glance, this seems reasonable, especially given how well Eq. (4) agrees with the data, but, in dropping the "quickly vanishing" part of Eq. (3), what they have effectively done is to *derive* an initial condition. This is unnecessary, since, when the bullet reaches the target, its yaw and yaw rate are each free to have any value, without regard to Eq. (5). The yaw and its rate can even have opposite signs, with the bullet yawed in one direction but rotating in the opposite direction (i.e., with yaw instantaneously decreasing).

This is seen by evaluating the general solution Eq. (3) and its derivative at the initial state,

$$\alpha_0 = \alpha_{01} + \alpha_{02} \tag{6}$$

$$\alpha'_0 = \sqrt{M}\alpha_{01} - \sqrt{M}\alpha_{02} \tag{7}$$

where the initial yaw rate $\alpha'_0 = d\alpha/ds$ at $s = s_0$. Choosing appropriate values of α_{01} and α_{02} can satisfy any set of initial conditions α_0 and α'_0. Roecker's omission of the "rapidly vanishing" second term of Eq. (3) forces its coefficient α_{02} to be zero, which leads to Eq. (5).

If we choose instead $\alpha_{01} = 0$, then the *first* term of Eq. (3) is eliminated, and the "rapidly vanishing" second term constitutes the entire solution.

$$\alpha = \alpha_0 \exp\left[-\sqrt{M}(s - s_0)\right] \tag{8}$$

in which case, the yaw of the bullet tends not to increase unstably but rather to decrease—a condition of yaw stability! This solution has as its initial condition

$$\left.\frac{d\alpha}{ds}\right|_{s=s_0} = -\sqrt{M}\alpha_0 \tag{10}$$

which is the same as Eq. (5) but with a change in sign. Note that this state of stability is a singularly special case: any deviation from initial condition Eq. (10) results in $\alpha_{01} \neq 0$, thus activating the unstable first term of the general solution Eq. (3).

There is thus no reason that Roecker's initial condition Eq. (5) is required. Admittedly, Roecker intended this condition to apply to the instant when the yaw begins to grow unstably. This issue will be addressed with some further analysis.

ANALYSIS OF NON-LINEAR EQUATION

Some analysis of Roecker's governing equation, Eq. (1), sheds some light on its behavior. By multiplying by $(d\alpha/ds)ds$, it may be integrated exactly to become

$$\frac{1}{2}\left(\frac{d\alpha}{ds}\right)^2 = \frac{1}{2}M\sin^2\alpha + C_1 \tag{12}$$

or

$$\frac{d\alpha}{ds} = \pm\sqrt{M\sin^2\alpha + C_2} \tag{13}$$

where the sign is dictated by the initial value of yaw rate $d\alpha/ds$. It is however possible (depending on initial conditions) for the sign to later change.

The general solution of Eq. (13) involves an elliptic integral and cannot be cast as an explicit expression for α. However, for the special case $C_2 = 0$, it simplifies to

$$\frac{d\alpha}{ds} = \pm\sqrt{M}\,\sin\alpha \tag{14}$$

which requires either of the initial conditions (differing in sign)

$$\left.\frac{d\alpha}{ds}\right|_{s=s_0} = \pm\sqrt{M}\,\sin\alpha_0 \tag{15}$$

which is similar to Roecker's initial condition, Eq. (5) above, with $\pm\sin\alpha_0$ replacing α_0. The sign in Eq. (14) follows that taken in the initial condition Eq. (15). Equation (14) is integrated to

$$\ln\tan\frac{\alpha}{2} = \pm\sqrt{M}\,(s - s_0) + C_3 \tag{16}$$

where the constant depends on the initial yaw, $C_3 = \ln\tan(\alpha_0/2)$. The solution is

$$\alpha = 2\arctan\left\{\exp\left[\pm\sqrt{M}\,(s - s_0)\right]\tan\frac{\alpha_0}{2}\right\} \tag{18}$$

This equation with plus sign is plotted in Fig. 1, along with Roecker's approximate solution, Eq. (4) above, for $\sqrt{M} = 0.165$ and $\alpha_0 = 0.085$ radian, as well as Roecker's data. The curves agree closely with those in Roecker's paper, for which values of M and α_0 were not given.

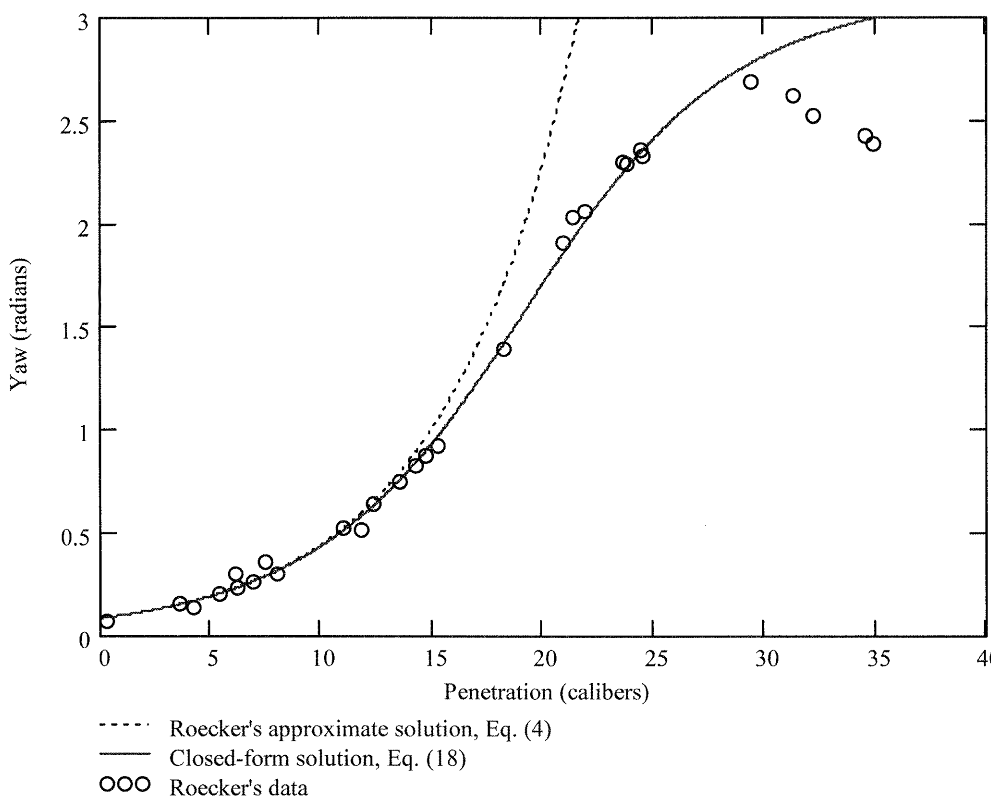

Fig. 1. Yaw vs. penetration path length according to Eqs. (4) and (18).

Eq. (18) may also be written as

$$\tan\frac{\alpha}{2} = \tan\frac{\alpha_0}{2}\exp\left[\pm\sqrt{M}(s-s_0)\right] \tag{19}$$

from which it is clear that both solutions (either sign) are asymptotic: as s increases, then $\tan(\alpha/2)$ approaches 0 for the minus sign and ∞ for the plus sign,

$$\text{As } s \to \infty, \tan\frac{\alpha}{2} \to \begin{cases} 0 \text{ for } \alpha_0' = -\sqrt{M}\alpha_0 \\ \infty \text{ for } \alpha_0' = +\sqrt{M}\alpha_0 \end{cases} \text{ or } \alpha \to \begin{cases} 0 \text{ for } \alpha_0' = -\sqrt{M}\alpha_0 \\ \pi \text{ for } \alpha_0' = +\sqrt{M}\alpha_0 \end{cases}$$

Thus, initial condition Eq. (15) with plus sign (which we might call a "Roecker-like initial condition") soon leads to a reversal in the penetrator's orientation, followed by stable behavior with the yaw approaching π. On the other hand, Eq. (15) with minus sign leads to a stable trajectory with the yaw approaching zero.

The former behavior, where the penetrator reverses orientation then penetrates stably, is called "toppling with axial stability" (Halsey and Austin, 1974, quoting a 1949 British Admiralty textbook); it has been observed for bullets penetrating soil targets. While for real bullets such behavior might be due to other factors (such as the bullet's front-to-back asymmetry, it is interesting that the model predicts it also in this special case where M is always positive and independent of α.

For a small angle α, which is approximately equal to its tangent, Eq. (19) may be written as

$$\alpha = \alpha_0 \exp\left[\pm\sqrt{M}(s-s_0)\right] \tag{20}$$

where taking either sign results in either Eq. (4) or Eq. (9). This clarifies the observation that Eq. (18) may, early in the penetration, behave either stably or unstably depending on the initial conditions, specifically on which sign is taken in Eq. (15).

Choosing the minus in Eq. (18), which corresponds to initial condition Eq. (15) also with minus, results in a stable solution,

$$\alpha = 2\arctan\left\{\exp\left[-\sqrt{M}(s-s_0)\right]\tan\frac{\alpha_0}{2}\right\} \tag{21}$$

As with the stable solution Eq. (9) of the linearized equation, this stable solution of the non-linear equation must be singular: any deviation, however small, of the initial condition from Eq. (15) will incite unstable yaw growth.

Note that initial condition Eq. (15) is the same as differential equation (14) at a specific position; i.e., the proportionality of yaw rate to yaw given by the initial condition holds at each position. This must be so because the conditions at any instant are the initial conditions thenceforward; to sustain stability, the relationship expressed by the initial condition for stability must continue to hold at every instant.

The stable initial condition involves the constant M of the differential equation. Thus, any physical factor that even infinitesimally alters the value of M, such as inhomogeneity of the target or erosion of the penetrator, even if perfectly symmetric, is sufficient to upset the condition of yaw stability. Thus, we cannot expect to observe yaw stability, except perhaps for a short time.

NUMERICAL SOLUTION OF NON-LINEAR EQUATION

A Runge-Kutta solution of Eq. (1) has been implemented in Mathcad. The scheme handles any initial conditions of yaw α_0 and yaw rate α'_0. The scheme was verified to accurately reproduce the special-case solutions, Eqs. (18) and (21).

Using this scheme, the model's behavior may be examined for other sets of initial conditions, all involving small or zero initial yaw. Figure 2 shows that, for initial yaw $\alpha_0 = 0.085$ but zero rate $\alpha'_0 = 0$, the yaw, as expected, takes longer to develop than for the Roecker-like initial conditions [Eq. (15) with plus sign], penetrating about another 3.5 calibers to reach the same value of yaw.

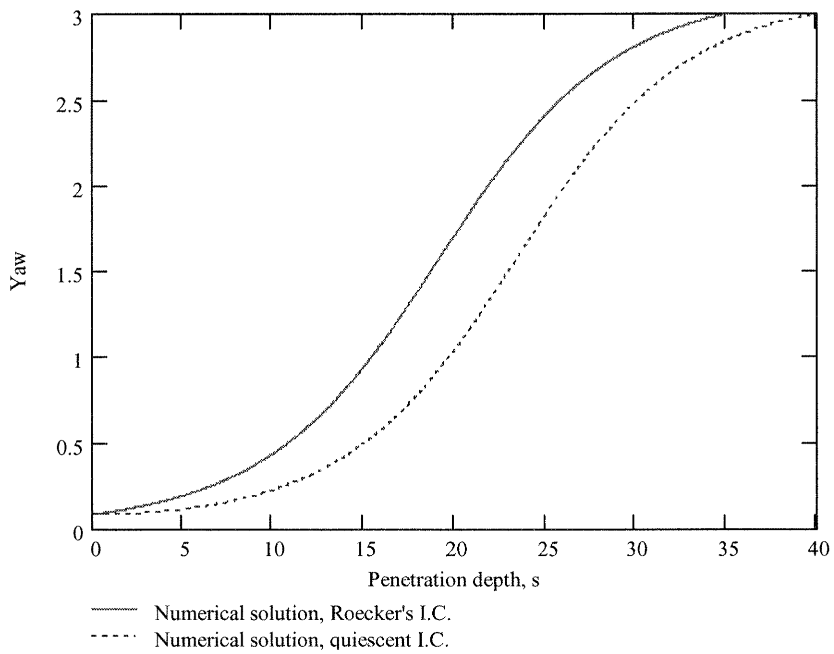

Fig. 2. Numerical solutions for zero-yaw-rate and Roecker-like initial conditions.

The solution for nearly stable initial conditions, similar to "stable" conditions [Eq. (15) with minus] but with an initial yaw rate that is 0.1% slower, $\alpha'_0 = -(0.999)\sqrt{(M)}\sin\alpha_0$, is compared with the result for the stable initial conditions in Fig. 3. The nearly stable solution deviates visibly from the stable solution at a depth of about 15 calibers, after which its yaw grows increasingly rapidly.

Another interesting case is an initial yaw rate larger than that given by the Roecker-like condition of Eq. (15) with plus sign. Figure 4 shows the solution for an initial yaw rate equal to twice this value. In this case, the yaw increases monotonically, with the penetrator continuously tumbling end-over-end in one direction, at a rate that oscillates between small and large values.

Computed results for seven sets of initial conditions are compared in Fig. 5. The various curves show different tendencies toward instability, in terms of apparent onset of rapid yaw growth. The solution for double the initial yaw rate of the Roecker-like condition is most unstable, followed by the Roecker-like, zero-yaw-rate, half the stable rate, then a tie between the nearly stable rate and a just-greater-than-stable rate, followed by the stable initial conditions. By a depth of ~170 calibers, even this last theoretically stable solution has actually turned unstable, which is attributed to the computer's finite accuracy; some deviation from the stable condition eventually creeps into the computation, and yaw begins to grow.

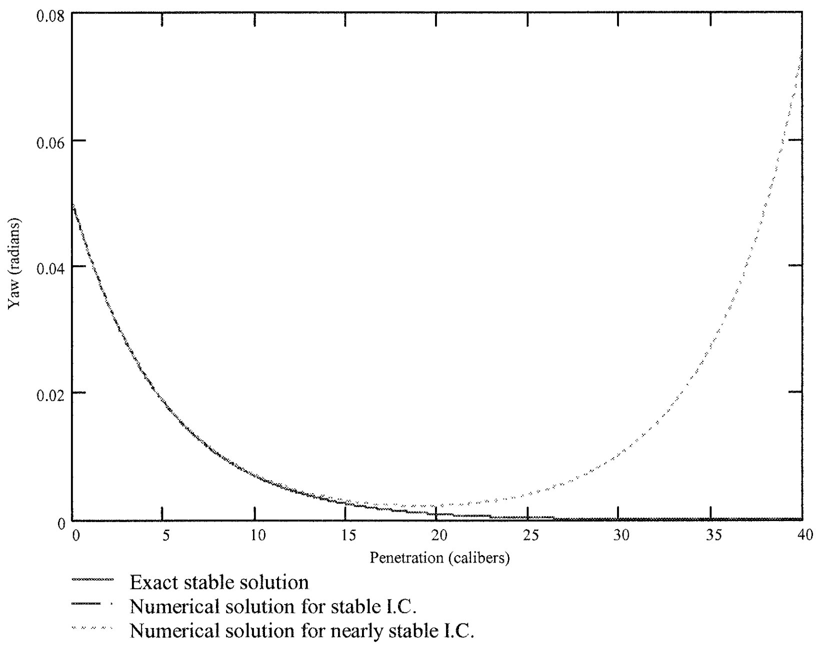

Fig. 3. Numerical solutions for stable and nearly stable initial conditions.

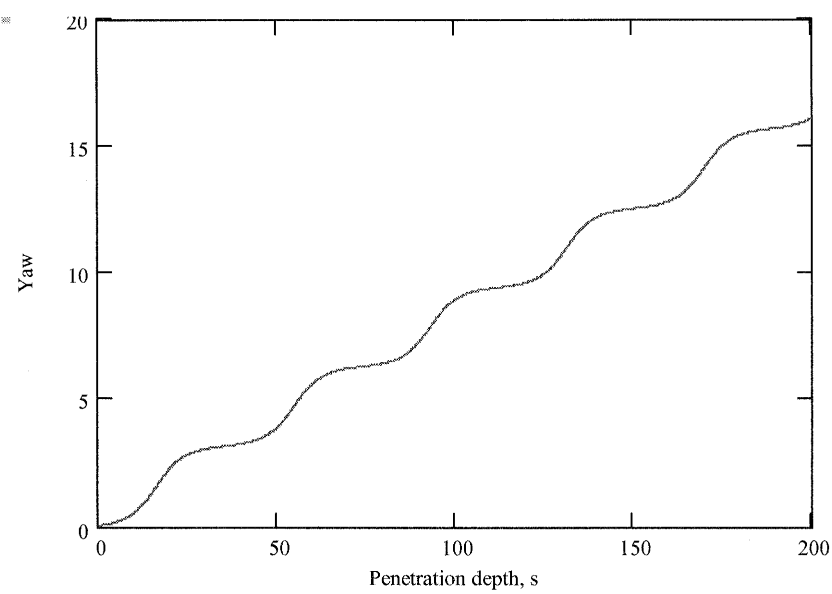

Fig. 4. Numerical solution for initial yaw rate twice that given by the Roecker-like initial conditions.

Figure 5 also shows that the solution for the Roecker-like initial conditions quickly achieves a large yaw (α near π radians, so that the penetrator is traveling backwards), where it remains for a considerable path length—the behavior known as “toppling with axial stability.” The solution for zero yaw rate is periodic: the oscillation in yaw is nearly sinusoidal, with no dwell at $\alpha = 0$ or π.

In Fig. 6, these curves are shifted horizontally by different distances so that their rapidly rising parts coincide (the curve for Roecker-like initial conditions is left unshifted). The close similarity among these curves demonstrates the validity of the method used by Roecker to shift his experimental data onto a single curve.

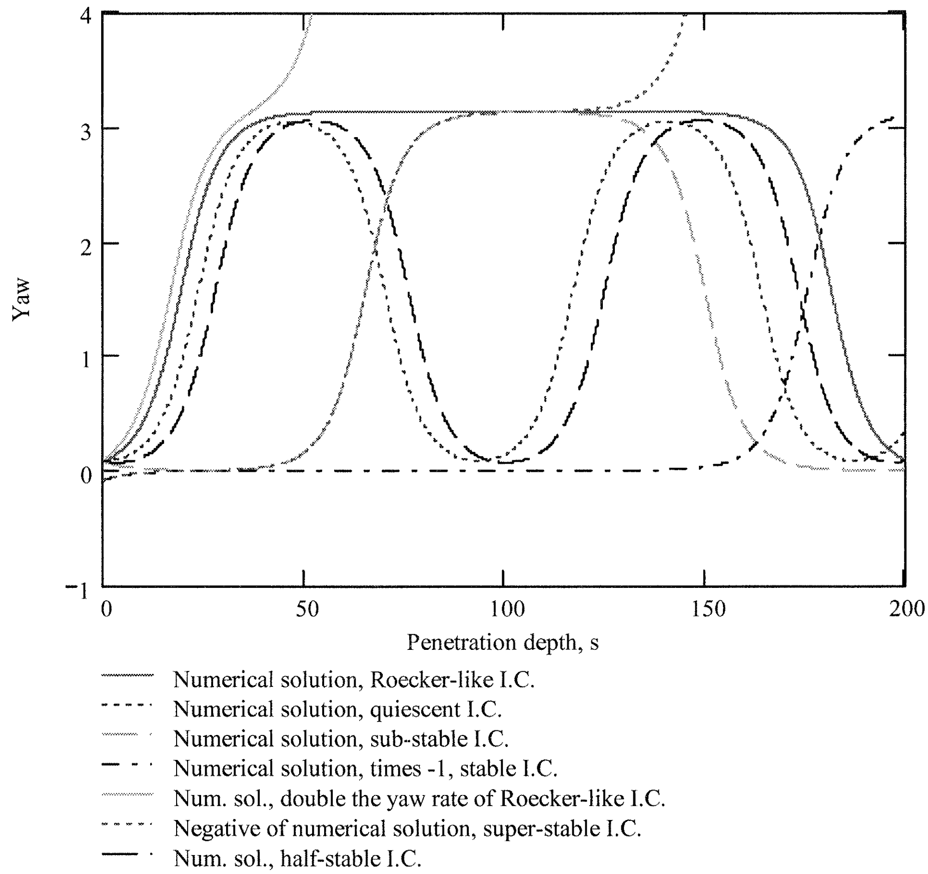

Fig. 5. Numerical solutions for seven sets of initial conditions.

CONCLUSIONS

Roecker and Ricchiazzi investigated the behavior of their model of yaw instability for only a limited set of initial conditions. By numerically solving their equation, we have been able to investigate other sets of initial conditions, which lead to a whole range of yaw behaviors:

- The yaw may continuously decrease (stable trajectory); however, this behavior, while mathematically possible, is physically unlikely.
- The penetrator may tumble (yaw in one direction) continuously.
- The penetrator may oscillate in orientation, front to back to front, etc.
- The penetrator may reverse orientation, then penetrate some distance apparently stably in that orientation.

It was also demonstrated that for small initial yaws and yaw rates, the parts of the solutions during which the yaw is noticeably increasing are all similar. By shifting the resulting yaw curves along the axis of penetration depth, they can all be made to coincide very closely.

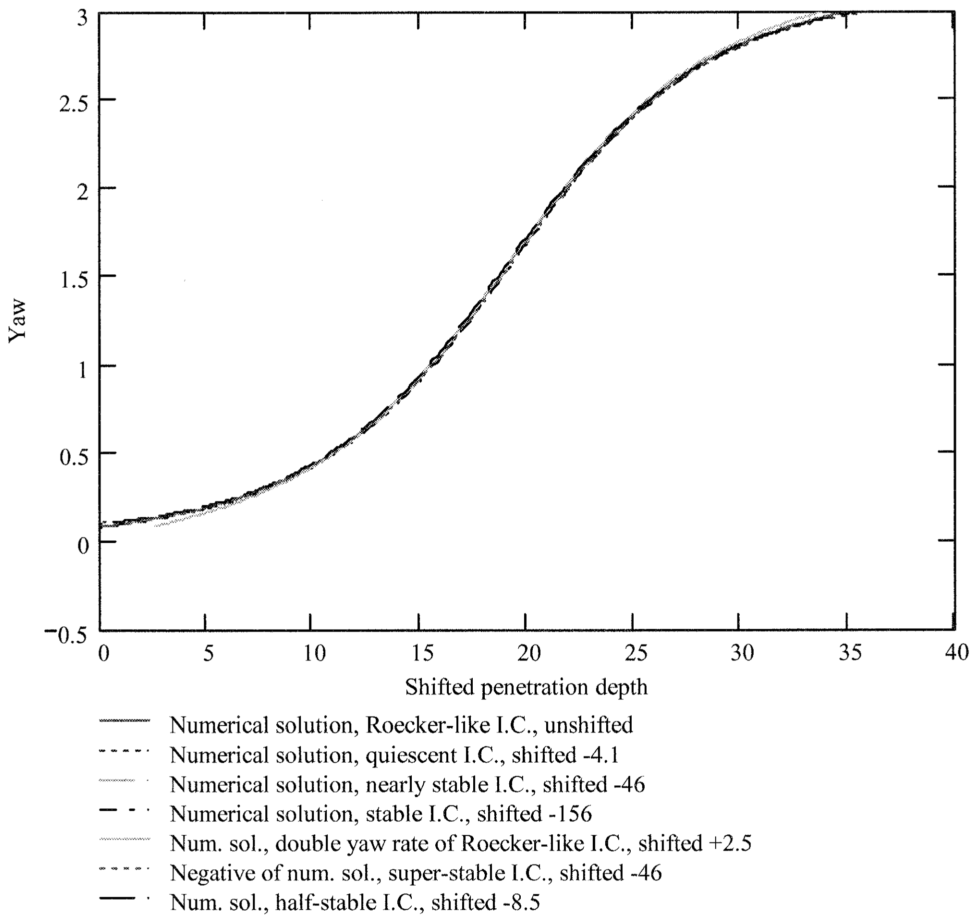

Fig. 6. Numerical solutions of Roecker's equation for seven sets of initial conditions, each shifted horizontally by a different distance so that all coincide.

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