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ANALYTICAL MODELS OF THE PROJECTION ANGLE OF EXPLOSIVELY ACCELERATED LINERS[1]

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dedicated to Prof. Pei Chi Chou, on the occasion of his 70th birthday

Explicit, analytical formulas are derived for the projection angle of explosively accelerated liners. The approach combines a dynamic model of liner acceleration by explosives with the unsteady Taylor-angle model of Chou et al. The resulting formulas are useful in predicting the performance of fragmenting and directed-energy warheads.

INTRODUCTION

The prediction of warhead performance requires knowledge not only of the liner's projection velocity V_0 but also its projection direction, or angle, δ_0. Taylor[2], in his analysis of the explosion of a cylindrical bomb, developed his well-known equation

$$\sin\delta_0 = \frac{V_0}{2D} \tag{1}$$

where D is the detonation velocity (replaced, more generally, by the detonation-wave sweep speed U). Taylor's analysis was predicated on a steady state, attainment of which requires the velocity and acceleration to be uniform along the liner. However, in many warheads, and near the ends of all warheads, the motion is not uniform, and the projection angle differs from that predicted by Eq. (1).

To account for this difference, Randers-Pehrson[3] developed the empirical relation

$$\sin\delta_0 = \frac{V_0}{2U} - \frac{1}{2}V'_0\tau_e - \frac{1}{5}(V'_0\tau_e)^2 \tag{2}$$

in which primes denote spatial derivatives along the liner (i.e., $V_0' = \partial V_0/\partial l$, where l is the coordinate along the liner) and where τ_e is a characteristic acceleration time obtained by fitting an exponential function to the liner's velocity history,

$$V(t) = V_0\left[1 - \exp\left(-\frac{t-T}{\tau_e}\right)\right] \tag{3}$$

where T is the time of onset of motion (i.e., arrival of the detonation wave).

Subsequently, Chou et al.[4,5] succeeded in deriving the general unsteady Taylor relation,

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[1] This work was supported by the Armament Division of Wright Laboratory, Eglin AFB, FL, under Contract F008630-94-C-0070. The guidance of Dr. Joseph C. Foster, Jr. is gratefully acknowledged.

[2] G.I. Taylor, "Analysis of the Explosion of a Long Cylindrical Bomb Detonated at One End," (1941), *The Scientific Papers of Sir G.I. Taylor*, Vol. III, G.K. Batchelor, ed., Cambridge, pp. 277-286, 1963.

[3] G. Randers-Pehrson, "An Improved Equation for Calculating Fragment Projection Angle," *Proc. 2nd ISB*, Daytona Beach, FL, 1976.

[4] P.C. Chou, J. Carleone, E. Hirsch, W.J. Flis, and R.D. Ciccarelli, "Improved Formulas for Velocity, Acceleration, and Projection Angle of Explosively Driven Liners," *Proc. 6th ISB*, Orlando, FL, 1981.

[5] P.C. Chou, E. Hirsch, and R.D. Ciccarelli, "An Unsteady Taylor Angle Formula for Liner Collapse," Ballistic Research Laboratory, Contract Report ARBRL-CR-00461, 1981. AD A104682.

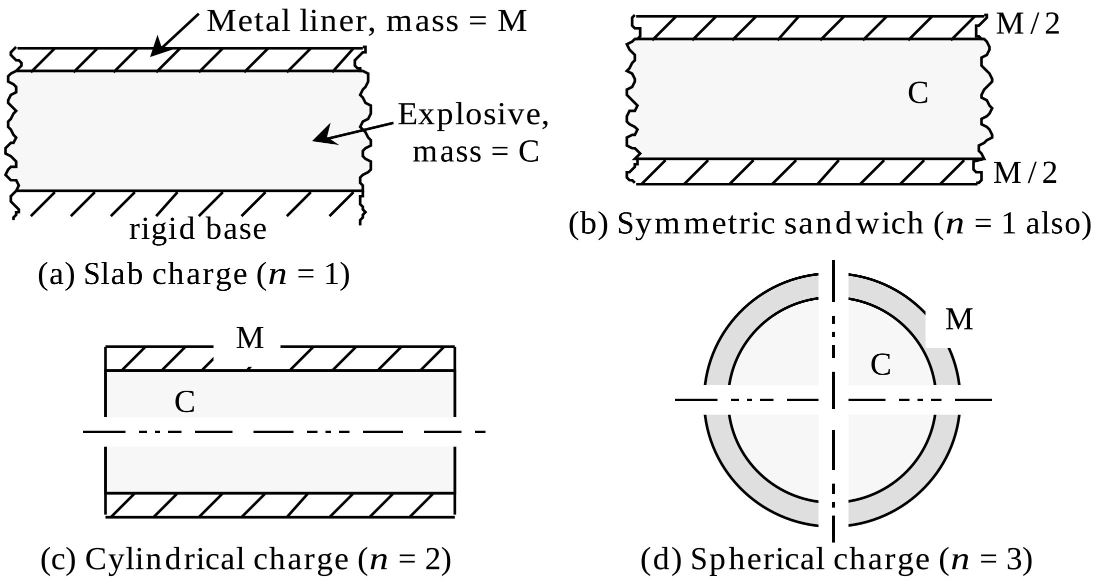


Fig. 1. Simple, single-degree-of-freedom (1-DOF) explosive-metal systems.

$$\delta_0 = \lim_{t \to \infty} \left(-\frac{\partial}{\partial l} \int_T^t V\,dt + \frac{1}{2V} \frac{\partial}{\partial l} \int_T^t V^2\,dt \right) \qquad (4)$$

Application of this relation to the exponential velocity history yielded the formula,

$$\delta_0 = \frac{V_0}{2U} - \frac{1}{2} V'_0 \tau_e + \frac{1}{4} V_0 \tau'_e \qquad (5)$$

While both unsteady Taylor formulas, Eqs. (2) and (5), have been shown to predict accurate values of δ_0, the velocity history on which they are based, Eq. (3), is strictly empirical, and a separate formula, also empirical, must be developed to supply the value of τ_e for each explosive-metal system.

Recently[6], a model of liner acceleration by explosive was developed using Lagrange's equation. For simple, single-degree-of-freedom (1-DOF) explosive-metal systems (Fig. 1), this model yielded the equation of motion,

$$\left(M + \frac{n}{n+2} C \right) \ddot{x} = \frac{n P_0 C}{\rho_0 x_0} \left(\frac{x_0}{x} \right)^{n(\gamma - 1) + 1} \qquad (6)$$

where P_0, ρ_0, and γ are the explosive's initial pressure, density, and specific-heat ratio, and n=1 for planar charges, n=2 for cylinders, and n=3 for spheres. This equation may be integrated to obtain the velocity $V(x)$ as a function of liner position x; for the special case n=1, γ=3, it may be integrated again to derive the explicit velocity history,

$$V(t) = \dot{x} = V_0 \left[1 + \left(\frac{\tau_m}{t - T} \right)^2 \right]^{-1/2}, \text{ where } \tau_m = x_0 / V_0, \qquad (7)$$

An important aspect of this model is that it predicts not only the liner's final velocity V_0 but also its velocity history $V(t)$.

In this paper, unsteady Taylor formulas for various explosive-metal systems are analytically derived from velocity histories predicted by the model of Ref. 6, such as Eq. (7), for which the time constant τ_m has a simple definition common to all systems.

[6] W.J. Flis, "A Lagrangian Approach to Modeling the Acceleration of Metal by Explosives," *17th SouthEastern Conference on Theoretical and Applied Mechanics*, Hot Springs, AK, April 10-12, 1994.

MODEL FOR 1-DOF SYSTEMS

We first derive an unsteady Taylor formula based on the velocity history predicted by Eq. (6) for simple, single-degree-of-freedom systems. Now, Eq. (6) can be solved to describe the liner velocity as an implicit function of time,

$$\frac{t-T}{\tau_m} = \frac{1}{p}\, \mathrm{B}_{(V/V_0)^2}\left(\frac{1}{2}, -\frac{1}{p}\right) \tag{8}$$

where $\tau_m = x_0 / V_0$, $p = n(\gamma - 1)$, and B_x is the Incomplete Beta Function, defined as

$$\mathrm{B}_x(m, n) \equiv \int_0^x z^{m-1}(1-z)^{n-1} dz$$

Since this result describes the velocity history only in implicit terms (unless p=2, for which it reduces to Eq. (7)), it cannot be used directly in Eq. (4). However, it can still serve as the basis for deriving an unsteady Taylor formula.

Consider a velocity history of general form $V = V_0 f\{(t-T)/\tau\}$, where τ is a constant with units of time. (Thus, for Eq. (7), we have $f = (1+\eta^{-2})^{-1/2}$, where $\eta = (t-T)/\tau$, with $\tau = \tau_m$; and for Eq. (3), $f = 1 - e^{-\eta}$, with $\tau = \tau_e$.) For this general form, Eq. (4) reduces to

$$\delta_0 = \frac{V_0}{2U} + \left(I_1 - I_2\right) V'_0 \tau + \left(I_1 - \frac{1}{2} I_2\right) V_0 \tau' \tag{9}$$

where
$$I_1 = \int_0^\infty \left[1 - f(\eta)\right] d\eta \; ; \quad I_2 = \int_0^\infty \left[1 - f^2(\eta)\right] d\eta \tag{10}$$

For a given function f, the definite integrals I_1 and I_2 are constants. For example, for the exponential history, Eq. (3), these are easily evaluated as $I_1 = 1$ and $I_2 = 3/2$, insertion of which into Eq. (9) yields Eq. (5). More generally, Eq. (9) shows that derivation of an explicit unsteady Taylor formula requires not an explicit expression for f but only evaluation of the integrals I_1 and I_2. This can be done for Eq. (6) with some manipulation.

First, Eq. (6) is written in dimensionless form, by defining the dimensionless position $X = x/x_0$, and the dimensionless time $\eta = (t-T)/\tau_m$,

$$\frac{d^2 X}{d\eta^2} = \frac{p}{2} X^{-p-1}$$

Integrating once and solving for X yields

$$X = \left[1 - \left(\frac{dX}{d\eta}\right)^2\right]^{-\frac{1}{p}}$$

Differentiating this, then substituting $f = dX/d\eta$ yields a differential equation for f,

$$\frac{df}{d\eta} = \frac{p}{2}\left(1 - f^2\right)^{\frac{p+1}{p}}$$

(This indicates that f, a function of η, depends also on the parameter $p = n(\gamma - 1)$, assumed constant.) By this equation, we can substitute for $d\eta$ in the integrals to get, e.g.,

$$I_2 = \int_0^\infty \left[1 - f^2(\eta)\right] d\eta = \frac{2}{p} \int_0^1 \left[1 - f^2\right] \frac{df}{\left(1 - f^2\right)^{\frac{p+1}{p}}}$$

The substitution $f = \sin\theta$ yields

$$I_2 = \frac{2}{p}\int_0^{\pi/2} (\cos\theta)^{1-\frac{2}{p}}\, d\theta = \frac{1}{p}\,\mathrm{B}\left(\frac{1}{2}, 1-\frac{1}{p}\right) = \frac{\sqrt{\pi}}{p}\,\frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{p}\right)} \tag{11}$$

where B and Γ are the well-known Beta and Gamma Functions. Similarly,

$$I_1 = \left(1-\frac{p}{2}\right)I_2 + 1 \tag{12}$$

These two formulas, with Eq. (9), constitute a completely derived, unsteady Taylor formula for simple, 1-DOF systems. For the special case of n=1 and γ=3 (i.e., p=2) with the explicit velocity history Eq. (7), these formulas give values of $I_1 = 1$ and $I_2 = \pi/2$, so that

$$\delta_0 = \frac{V_0}{2U} - \left(\frac{\pi}{2} - 1\right) V'_0\tau_m + \left(1-\frac{\pi}{4}\right) V_0\tau'_m$$

This is similar to Chou et al.'s formula for the exponential velocity history, Eq. (5), except for the values of the coefficients and the replacement of τ_e by τ_m.

Now, Eq. (7) may be written in the form

$$\delta_0 = \frac{V_0}{2U} - C_1 V'_0\tau_m + C_2 V_0\tau'_m \tag{13}$$

The coefficients, $C_1 = I_2 - I_1$ and $C_2 = I_1 - I_2/2$, are plotted over the possible range of values of n and γ (or p) in Fig. 2. For condensed explosives, the ratio of specific heats of the product gases varies over the range $2.5 \le \gamma \le 3.5$. Therefore, for planar charges (n=1), the parameter $p = n(\gamma - 1)$ can have values of 1.5 to 2.5; for cylinders (n=2), 3.0 to 5.0; and for spheres (n=3), 4.5 to 7.5, as indicated in the figure.

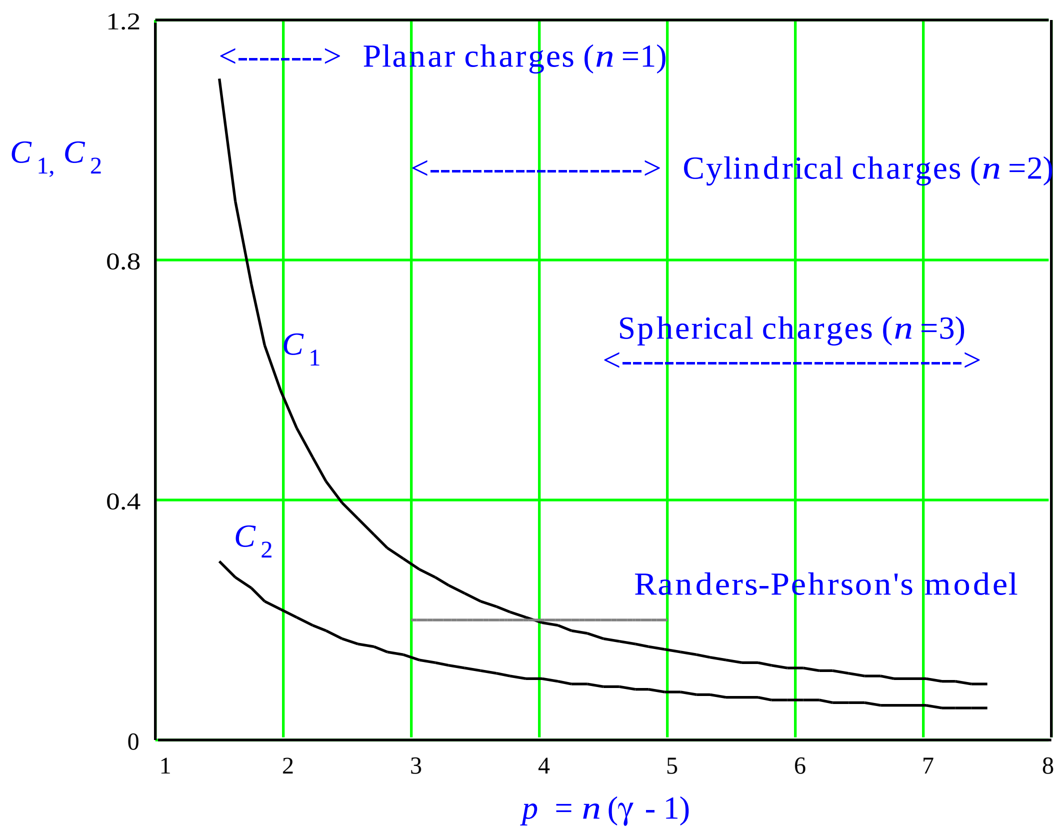


Fig. 2. Plot of constants C_1 and C_2 for 1-DOF systems versus the parameter p.

This figure also compares the present model with the empirical model for cylindrical charges of Randers-Pehrson (Ref. 3), who combined the formula

$$\tau_e = \frac{0.4\, x_0}{V_0} \tag{14}$$

with his empirical unsteady Taylor formula, Eq. (2). Since, for small angles, sin $\delta_0 \approx \delta_0$, and since $\tau_m = x_0 / V_0$, Eq. (2) may be written as

$$\delta_0 = \frac{V_0}{2U} - 0.2 V'_0 \tau_m - \frac{0.4^2}{5} (V'_0 \tau_m)^2$$

Thus, Randers-Pehrson's model corresponds to a value of $C_1 = 0.2$ in Eq. (13). This value, plotted in Fig. 2 over the range of p for cylinders, lies across the C_1 curve of the present model, but does not reflect any effect of the value of γ. (The last term in Eq. (2) differs in form from that in Eq. (13), so there is no corresponding C_2 term. It may be recognized that, in both equations, the contributions of the last terms are small.)

This agreement is explained by the very good agreement of the exponential history with the prediction of Eq. (6) for cylindrical charges. Figure 3(a) is a plot of velocity histories computed from Eq. (6) with n=2 for several values of γ (these are also plots of $f(\eta)$). These curves may be brought nearly into coincidence by scaling the time by a factor involving γ, shown in Fig. 3(b) (the factor chosen has the effect of equalizing the initial slopes). Figure 3(b) also includes a curve of the exponential history, Eq. (3), with $\tau_e = \frac{0.8\, x_0}{(\gamma - 1) V_0}$, which is proposed as a modification of Eq. (14) to account for the value of γ. This curve differs from the model predictions by, at most, 2.8% of V_0.

For planar charges, agreement between the exponential function and the model is not as good. A better approximation is to use Eq. (7) with a modified characteristic time, $\tau_m^* = \frac{2\, x_0}{(\gamma - 1) V_0}$, which is exact for $\gamma = 3$ and is very close for other values of γ.

APPLICATION TO OTHER SYSTEMS

The same approach is applicable to any explosive-metal system to which the Gurney model is applicable. Here it is applied to several other important systems.

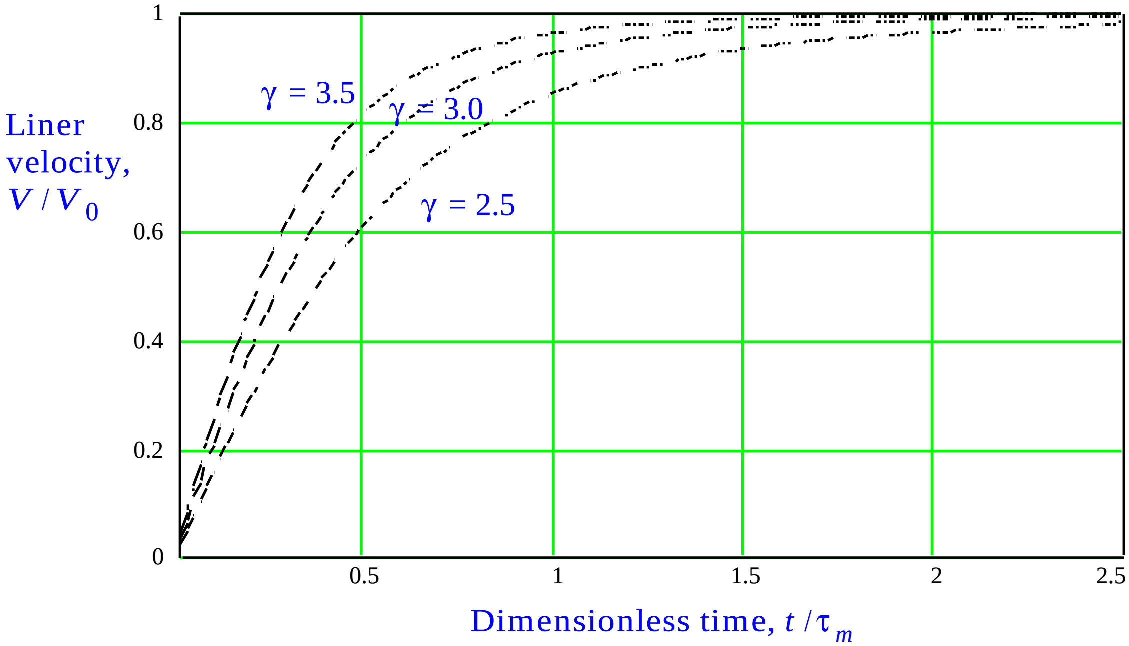
Asymmetric Sandwich

The above derivation is directly applicable to the asymmetric sandwich. Ref. 6 gives its equations of motion,

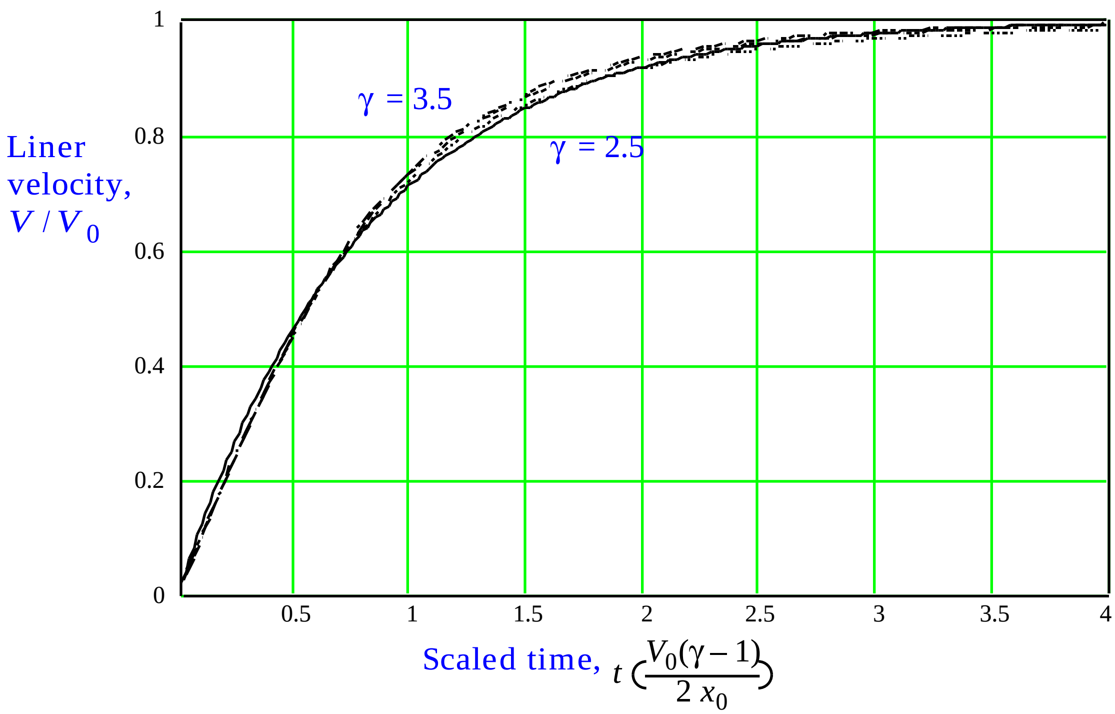
$$\begin{aligned} \left(M_1 + \frac{C}{3}\right) \ddot{x}_1 - \frac{C}{6} \ddot{x}_2 &= P_0 \left(\frac{x_0}{x_o + x_1 + x_2}\right)^\gamma \\ -\frac{C}{6} \ddot{x}_1 + \left(M_2 + \frac{C}{3}\right) \ddot{x}_2 &= P_0 \left(\frac{x_0}{x_o + x_1 + x_2}\right)^\gamma \end{aligned} \tag{15}$$

Since the right-hand sides are identical, $\left(M_1 + \frac{C}{3}\right) \ddot{x}_1 - \frac{C}{6} \ddot{x}_2 = -\frac{C}{6} \ddot{x}_1 + \left(M_2 + \frac{C}{3}\right) \ddot{x}_2$

Re-arrangement and integration yields $\left(M_1 + \frac{C}{2}\right) \dot{x}_1 = \left(M_2 + \frac{C}{2}\right) \dot{x}_2$, which is equivalent to conservation of momentum, as used in the Gurney model for this system. Since this holds



(a) Velocity histories versus dimensionless time



(b) Velocity histories versus scaled time

Fig. 3. Liner velocity histories for cylindrical charges, as computed from the dynamic model, Eq. (6) (dashes), and as given by the exponential function, Eq. (3) (solid curve).

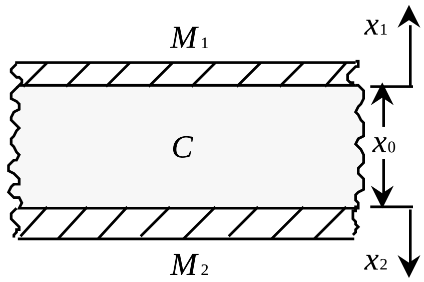


Fig. 4. Asymmetric sandwich.

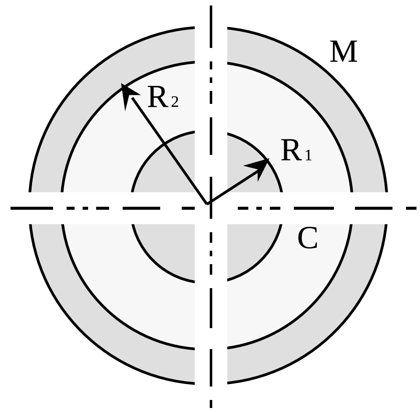


Fig. 5. Cylindrical charge with rigid core.

at all times, both velocity histories must follow the same function, i.e.,

$$V_1(t) = \dot{x}_1 = V_{1,0}\, f\left(\frac{t-T}{\tau_m}\right) \quad \text{and} \quad V_2(t) = \dot{x}_2 = V_{2,0}\, f\left(\frac{t-T}{\tau_m}\right) \qquad (16)$$

which differ only in the final (Gurney) velocities, $V_{1,0}$ and $V_{2,0}$.

Now, it is shown in Ref. 6 that Eqs. (15) can be combined into

$$\frac{M_1 M_2 + \frac{1}{3} C (M_1 + M_2) + \frac{1}{12} C^2}{M_1 + M_2 + C}\, \ddot{\chi} = P_0 \left(\frac{x_0}{\chi}\right)^{\gamma}$$

where $\chi = x_1 + x_2 + x_0$. This is similar to Eq. (6), with x replaced by χ, $n = 1$, and a different inertia term; thus, it must have a solution with the same velocity function f. Finally, since χ is a linear combination of x_1 and x_2, this function must be the same as f in Eq. (16). Therefore, the unsteady Taylor formula for this system is Eq. (7) with constants given by Eqs. (11-12) with n=1 and $\tau_m = x_0/(V_{1,0} + V_{2,0})$.

Cylindrical Charge with Rigid Core

The present approach may be applied to a cylindrical charge having a rigid core. By the method of Ref. 6, its equation of motion is

$$\left[M + \frac{C}{6}\,\frac{R_1 + 3R_2}{R_1 + R_2}\right]\ddot{r} = 2\pi P_0 \left(\frac{R_2^2 - R_1^2}{r^2 - R_1^2}\right)^{\gamma} r$$

By a similar procedure to that above, the differential equation for f may be derived:

$$\frac{df}{d\eta} = \frac{\gamma - 1}{1 - a^2}\sqrt{1 - a^2 + a^2(1 - f^2)^{\frac{1}{\gamma - 1}} - (1 - f^2)^{\frac{2\gamma - 1}{2\gamma - 2}}} \qquad (17)$$

where we have used $\tau_m = R_2/V_0$ and $a = R_1/R_2$. Note that, in this case, the function f depends not only on γ but also on a, which could, of course, vary along a liner (as for a non-uniform core radius); then, I_1 and I_2 would also vary along the liner, and the unsteady Taylor formula must include additional terms,

$$\delta_0 = \frac{V_0}{2U} + (I_1 - I_2)V'_0\tau_m + (I_1 - \tfrac{1}{2}I_2)V_0\tau'_m + (I'_1 - \tfrac{1}{2}I'_2)V_0\tau_m \qquad (18)$$

By Eq. (17), the integrals I_1 and I_2 may be written as

$$I_1 = \frac{1-a^2}{\gamma-1}\int_0^1 \frac{(1-f)(1-f^2)^{-\frac{2\gamma-1}{2(\gamma-1)}}\,df}{\sqrt{1-a^2+a^2(1-f^2)^{\frac{1}{\gamma-1}}}} \quad \text{and} \quad I_2 = \frac{1-a^2}{\gamma-1}\int_0^1 \frac{(1-f^2)^{-\frac{1}{2(\gamma-1)}}\,df}{\sqrt{1-a^2+a^2(1-f^2)^{\frac{1}{\gamma-1}}}} \qquad (19)$$

which, given values of γ and a, are constants. These can be evaluated numerically, as plotted in Fig. 6. Thus, the unsteady Taylor formula for this system is Eq. (18) with constants given by Eq. (19).

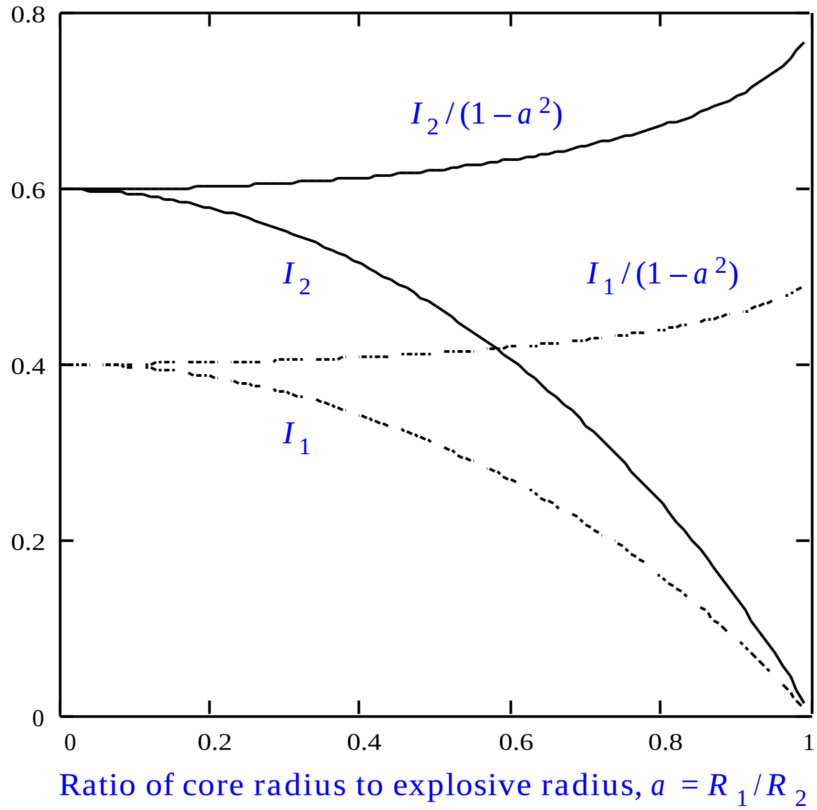


Fig. 6. Plot of integrals I_1 and I_2 for cored cylindrical charge vs. radius ratio a for $\gamma = 3$.

Hollow Cylindrical Charge

Chanteret[7] developed a Gurney-type model for the hollow cylinder. His model involves an additional assumption, that within the explosive exists a fictive rigid surface which never moves. The resulting formula gives accurate predictions of final velocities, and Ref. 6 shows that his assumption is in good agreement with solutions of equations of motion derived by the same approach as Eq. (6).

By recognizing that, with respect to the outer liner, this rigid surface is equivalent to a rigid core, we can directly adapt the above equations by substituting the rigid surface's radius R_x for R_1, so now $a = R_x / R_2$. Chanteret gives the equation for R_x, which, neglecting the direction of the detonation wave, may be written as

$$R_x^3 + 3\left[(R_1+R_2)\left(R_1\frac{M_2}{C} + R_2\frac{M_1}{C}\right) + R_1R_2\right]R_x - 3(R_1+R_2)R_1R_2\left(\frac{2}{3}+\frac{M_1}{C}+\frac{M_2}{C}\right) = 0 \qquad (20)$$

[7] P.Y. Chanteret, "An Analytical Model for Metal Acceleration by Grazing Detonation," *Proc. 7th ISB*, The Hague, The Netherlands, 1983.

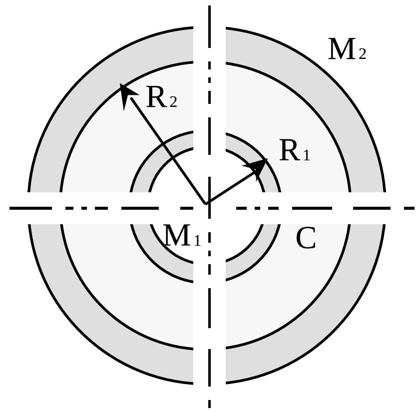


Fig. 7. Hollow cylindrical charge.

Therefore, the unsteady Taylor formula for the outer liner of this system is Eq. (18) with constants given by Eq. (19) with $a = R_x / R_2$ and R_x given by Eq. (20).

CONCLUSION

Unsteady Taylor formulas have been analytically derived for predicting the metal projection angle for several explosive-metal systems. These formulas account for unsteady effects of non-uniform velocity and acceleration along the metal liner, and are useful in the analysis of explosive warheads and reactive armor.

The approach is based on the application of the unsteady projection-angle model of Chou et al. to the velocity history given by a previously developed dynamical model of metal acceleration by explosives. The approach is applicable to all the same systems as the Gurney model and incorporates similar assumptions. For cylindrical charges, the present formula is shown to agree well with the empirical model of Randers-Pehrson, and has the advantage of also accounting for the equation of state of the explosive.